Compressible primitive equation: formal derivation and stability of weak solutions

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Abstract

We present a formal derivation of a simplified version of Compressible Primitive Equations (CPEs) for atmosphere modeling. They are obtained from 3-D compressible Navier-Stokes equations with an *anisotropic viscous stress tensor* where viscosity depends on the density. We then study the stability of the weak solutions of this model by using an intermediate model, called model problem, which is more simple and practical, to achieve the main result.

Keywords: Compressible primitive equations, Compressible viscous fluid, *a priori* estimates, Stability of weak solutions.

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1 Introduction

Among equations of geophysical fluid dynamics (see [8]), the equations governing the motion of the atmosphere are the Primitive Equations (PEs). In the hierarchy of geophysical fluid dynamics models, they are situated between non hydrostatic models and shallow water models. They are obtained from the full 3 dimensional set of Navier-Stokes equations for atmosphere modeling,

$$\rho\frac{D}{Dt}\mathbf{U} + \nabla p + \rho\mathbf{g} = D, \tag{1}$$

$$\frac{D}{Dt}\rho + \rho\mathrm{div}\mathbf{U} = 0, \tag{2}$$

$$c_p\frac{D}{Dt}T - \frac{1}{T}\frac{D}{Dt}p = Q_T, \tag{3}$$

$$\frac{D}{Dt}q = Q_q, \tag{4}$$

$$p = RT\,\rho \tag{5}$$

where

$$\frac{D}{Dt} = \partial_t + \mathbf{U}\cdot\nabla\,.$$

$\mathbf{U}$ is the three dimensional velocity vector with component $\mathbf{u}$ for horizontal velocity and v for the vertical one. The terms ρ, p, T, $\mathbf{g}$ stand for the density, the pressure, the temperature, the gravity vector $(0, 0, g)$. The diffusion term D is written as:

$$D = \mu\Delta_x\mathbf{U} + \nu\partial^2_{yy}\mathbf{U} \tag{6}$$

where Δ_x stands for the derivatives of second order with respect to the horizontal variables $x = (x_1, x_2)$, and $\mu \neq \nu$ represents the anisotrope pair of viscosity. The diffusive term Q_q represent the molecular diffusion where q is the amount of water in the air, and Q_T is the heat diffusion standing for the solar heating (see for instance [17] for details of diffusive terms). The last term c_p is the specific heat of the air at constant pressure and R is the specific gas constant for the air.

A scale analysis show that only the terms $\partial_y p$ and $g\rho$ are dominant (see e.g. [15]). This leads to replace the third equation of (1) with the hydrostatic one to obtain the so-called Compressible Primitive Equations (CPEs) for atmosphere modeling,

$$
\left\{
\begin{array}{l}
\rho \dfrac{d}{dt}\mathbf{u} + \nabla_x p = D,\\
\partial_y p = -g\rho,\\
\dfrac{d}{dt}\rho + \rho \mathrm{div}\mathbf{U} = 0,\\
c_p \dfrac{D}{Dt}T - \dfrac{1}{T}\dfrac{D}{Dt}p = Q_T,\\
\dfrac{D}{Dt}q = Q_q,\\
p = RT\,\rho
\end{array}
\right.
\tag{7}
$$

where x and y stand for the horizontal and vertical coordinate and

$$
\frac{d}{dt} = \partial_t + \mathbf{u}\cdot\nabla_x + v\partial_y\,.
$$

Derivation "background" In this paper, we present the derivation of Compressible Primitive Equations (CPEs) close to Equations (7) (without taking in account complex phenomena such as the amount of water in the air and the solar heating) from the 3-D Navier-Stokes equations with an *anisotropic viscous tensor*. Emphasizing to the difference of sizes of the vertical and horizontal dimensions in the atmosphere (10 to 20 for height with respect to thousands of kilometers of length), we derive the hydrostatic balance approximation for the vertical motion. We obtain *simplified CPEs*:

$$
\left\{
\begin{array}{l}
\partial_t\rho + \mathrm{div}_x\left(\rho\,\mathbf{u}\right) + \partial_y\left(\rho v\right) = 0,\\
\partial_t\left(\rho\,\mathbf{u}\right) + \mathrm{div}_x\left(\rho\,\mathbf{u}\otimes\mathbf{u}\right) + \partial_y\left(\rho\, v\mathbf{u}\right) + \nabla_x p(\rho) = \mathrm{div}_x\left(\nu_1 D_x(\mathbf{u})\right)\\
+\partial_y\left(\nu_2\partial_y\mathbf{u}\right),\\
\partial_y p(\rho) = -g\rho
\end{array}
\right.
\tag{8}
$$

where y stands for the vertical coordinate. The main difference between Model (7) and Model (8) is the viscous term. Moreover, if $p = c^2\rho$ with $c = RT$ (for instance, as above), the density ρ is written $\xi(t,x)e^{-g/c^2 y}$ where ξ , called again "density", is an unknown of the following system called *model problem*:

$$
\left\{
\begin{array}{l}
\dfrac{d}{dt}\xi + \xi\mathrm{div}_x(\mathbf{u}) + \xi\partial_z w = 0,\\
\xi\dfrac{d}{dt}\mathbf{u} + c^2\nabla_x(\xi) = D,\\
\partial_z\xi = 0
\end{array}
\right.
\tag{9}
$$

which is obtained from System (8) by the "simple" change of variables $z = 1 - e^{-y}$ and $w = e^{-y}v$ where

$$
\frac{d}{dt} = \partial_t + \mathbf{u}\cdot\nabla_x + w\partial_z
$$

and D stands for the following viscous terms

$$
D = \mathrm{div}_x\left(\nu_1 D_x(u)\right) + \partial_z\left(\nu_2\partial_z u\right).
\tag{10}
$$

As explained below, we cannot obtain a result directly on System (8) without using the intermediate Model (9), so we use the fact that System (9) is very close to System (8), and that the equation $\partial_z \xi = 0$ is one of the key ingredient to achieve the stability of weak solutions of System (9), to propagate the result to System (8).

Mathematical "background" The mathematical study of PEs for atmosphere modeling were first studied by J.L. Lions, R. Temam and S. Wang ([12]) where they produced the mathematical formulation of System (7) in 2 and 3 dimensions based on the works of J. Leray and they obtained the existence of weak solutions for all time (see also [17] where the result was proved by different means). For instance, in [17], using the hydrostatic equation, they used the pressure p as vertical coordinate instead of the altitude y. Moreover, they wrote System (7) in spherical coordinates (ϕ, θ, p) to change compressible equations to incompressible ones to use the well-known results of incompressible theory. They distinguished the *prognostic* variables from the *diagnostic* variables, which are: $(\mathbf{u}, T, q)$ for the prognostic and (v, ρ, Φ) for the diagnostic variables where Φ is the geopotential $gy(\phi, \theta, p, t)$. Diagnostic variables $(v = v(\mathbf{u}), \rho = \rho(T), \Phi = \Phi(T))$ can be written as a function of the prognostic variables through the div-free equation, $p = RT\rho$ and by integrating the mass equation which is written in the new coordinates as follows:

$$\partial_p \Phi + \frac{RT}{p} = 0.$$

Then, the outline of their proof of the existence was: they wrote

- a weak formulation of the PEs (by defining appropriate space functions) of the form
$$\frac{dU}{dt} + AU + B(U,U) + E(U) = l$$
where $U = (\mathbf{u}, T, q)$ with initial data $U(0) = U_0$ and A, B, E are appropriate functional,
- finite differences in time: U^n,
- *a priori estimates* for U^n,
- approximate functions: $U_{\Delta t}(t) = U^n$ on $((n-1)\Delta t, n\Delta t)$ (following [16])
- *a priori estimates* for $U_{\Delta t}$,

and they proved the passage to the limit.

Setting T and q constant, the main difference between Model (7) and (8) comes from the viscous term (6) and (10). Starting from the Navier-Stokes equations with *non constant density dependent viscosity* and *anisotropic viscous tensor*, it is natural to get the viscous term (10). This term is also present in the viscous shallow water equations (see e.g. [1]). In the same spirit than [17], authors [9] showed a global weak existence for the 2-D version of model problem (8) with $p(\rho) = c^2\rho$ (with the notation above) by a change of vertical coordinates (but not on p as done in [17]) which led to prove that Model (8) and Model (9) are equivalent. Then, using an existence result provided by Gatapov *et al* [10] for Model (9), they could conclude. Existence result [10] for System (9) was obtained as follows:

- by a useful change of variables in the Lagrangian coordinates, authors [10] showed that the density ξ is bounded from above and below .
- by *a priori* estimates and writing the system for the oscillatory part of the velocity $\mathbf{u}$, authors [10] obtained the existence result thanks to a Schauder fix point theorem.

Unfortunately, this approach [10] fails for the 3-D version (9) since the change of variables in Lagrangian coordinates does not provide enough information to bound the density ξ. Moreover, to show a stability result for weak solutions for Model (8) with standard techniques also fails, since multiplying the conservation of the momentum equations of Model (8) by (u, v) gives:

$$\frac{d}{dt}\int_\Omega \rho|u|^2 + \rho\ln\rho - \rho + 1\,dxdt + \int_\Omega \nu_1(\rho)|D_x(u)|^2 + nu_2(\rho)|\partial^2_{yy}u|\,dx + \int_\Omega \rho g v\,dx$$

where the sign of the integral $\displaystyle\int_\Omega \rho g v\,dx$ is unknown (in the equation above, $D_x(u)$ stands for $\dfrac{\nabla_x u + \nabla_x^t u}{2}$). It appears that, *prima facie*, there is a missing information on v to avoid the integral term $\displaystyle\int_\Omega \rho g v\,dx$ introduced by the hydrostatic equation $\partial_y p = -g\rho$. In fact, the study of the weak solutions stability cannot be performed directly on System (8) (at least up to our knownledge): therefore, we have to study the intermediate Model (9) through the change of vertical coordinates. Indeed, we have just remarked that (as described above) that ρ is written as $\xi e^{-g/c^2 y}$, where ξ does not depend on y. Thus, performing a change of variable in vertical coordinate in Model (8), following [9], we showed that Model (8) could be written as Model (9). As the hydrostatic equation in System (9) is $\partial_z\xi = 0$, an energy equality is easily obtained, and provided some *a priori* estimates. Nevertheless, those estimates are not strong enough to pass to the limit in the non linear terms; additional informations are required. On the other hand, we have to remark that the missing information for the vertical speed v for Model (8) (or equivalently w for Model (9)) is fulfilled by the equation of the mass of System (9), which is also written as:

$$\partial^2_{zz} w = \frac{1}{\xi}\mathrm{div}_x(\xi\partial_y \mathbf{u})\,.$$

Then, the fact that $\xi = \xi(t, x)$ combined with the equation above, allowed to obtain a mathematical entropy, the BD-entropy (initially introduced in [4], where a simple proof was given in [7, 6] or in [1] and the reference therein). Let us also notice that, as for shallow water equations (see e.g. [2, 3, 5] to cite only a few), it is necessary to add a regularizing term (as capillarity of friction) to equations (9) (equivalently to Model (8)) to conclude to the stability of weak solutions for Model (9): in this present case, we add a quadratic friction source term which is written $r\rho u|u|$ for System (8) or equivalently $r\xi u|u|$ for System (9). Indeed, the viscous term (10) combined to the friction term brings some regularity on the density, which is required to pass to the limit in the non linear terms (e.g. for the term $\rho u \otimes u$, where typically a strong convergence of $\sqrt{\rho}u$ is needed). Finally, by the reverse change of variables, the estimates, necessary to prove stability of weak solutions, were obtained for System (8) from those of System (9).

We note that, for the sake of simplicity, periodic conditions on the spatial horizontal domain Ω_x are assumed, since it avoids an incoming boundary term (whose sign is unknown: see e.g. [6]), which appears when we seek a mathematical BD-entropy. Let us also precise that "good" boundary conditions on Ω_x may be used (see [6]) instead of periodic ones to avoid this boundary term.

We may also perform this analysis without the quadratic friction term by using the "new" multiplier introduced in [13] which provides another mathematical entropy: particularly to estimate bounds of ρu^2 in a better space than $L^\infty(0,T;L^1(\Omega))$.

This paper is organized as follows In Section 2, starting from the 3-D compressible Navier-Stokes equations with an *anisotropic viscous tensor*, we formally derive the simplified Model (8) as described above. We present the main result in Section 2.2. In the third and last Section 3.2, we prove the main result. Firstly, we show that Model (8) can be rewritten as Model (9) which is more simpler. Then, taking advantages of the property of the density ξ, adding a quadratic friction term (following [2, 3]), we obtain a mathematical energy and entropy which provides enough estimates to pass to the limit in Model (9). Finally, following [9], the stability result for Model (8) is easily obtained.

2 Formal derivation of the simplified atmosphere model

We consider the Navier-Stokes model in a bounded three dimensional domain with periodic boundary conditions on Ω_x and free conditions on the rest of the boundary. More exactly, we assume that motion of the medium occurs in a domain $\Omega = \{(x,y);\, x \in \Omega_x,\, 0 < y < h\}$ where $\Omega_x = \mathbb{T}^2$ is a torus. The full Navier-Stokes equation is written:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{11}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\sigma(u) - \rho f = 0, \tag{12}$$

$$p = p(\rho) \tag{13}$$

where ρ is the density of the fluid and $u = (\mathbf{u}, v)^t$ stands for the fluid velocity with $\mathbf{u} = (u_1, u_2)^t$ the horizontal component and v the vertical one. The pressure law is given by the equation of state:

$$p(\rho) = c^2 \rho \tag{14}$$

for some given constant c. The term f is the quadratic friction source term and the gravity strength is given as follows:

$$f = -r\sqrt{u_1^2 + u_2^2}\,(u_1, u_2, 0)^t - g\mathbf{k}$$

where r is a positive constant coefficient, g is the gravitational constant and $\mathbf{k} = (0, 0, 1)^t$ (where X^t stands for the transpose of tensor X). The term $\sigma(u)$ is a non symmetric stress with the following viscous tensor (see e.g. [11, 10, 9]) $\Sigma(\rho)$:

$$\begin{pmatrix} \mu_1(\rho) & \mu_1(\rho) & \mu_2(\rho) \\ \mu_1(\rho) & \mu_1(\rho) & \mu_2(\rho) \\ \mu_3(\rho) & \mu_3(\rho) & \mu_3(\rho) \end{pmatrix}.$$

The total stress tensor is written:

$$\sigma(\mathbf{u}) = -pI_3 + 2\Sigma(\rho) : D(u) + \lambda(\rho)\mathrm{div}(u)\, I_3$$

where the term $\Sigma(\rho) : D(u)$ is written:

$$\begin{pmatrix} 2\mu_1(\rho) D_x(\mathbf{u}) & \mu_2(\rho)\,(\partial_y \mathbf{u} + \nabla_x v) \\ \mu_3(\rho)\,(\partial_y \mathbf{u} + \nabla_x v)^t & 2\mu_3(\rho)\partial_y v \end{pmatrix} \tag{15}$$

with I_3 the identity matrix. The term $D_x(\mathbf{u})$ stands for the strain tensor, that is: $D_x(\mathbf{u}) = \dfrac{\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}}{2}$ where $\nabla_x = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}$.

Remark 1 *Let us remark that, if we play with the magnitude of viscosity μ_i, the matrix $\Sigma(\rho)$ will be useful to set a privileged flow direction.*

The last term $\lambda(\rho)\mathrm{div}(u)$ is the classical normal stress tensor with $\lambda(\rho)$ the viscosity. The Navier-Stokes system is closed with the following boundary conditions on $\partial\Omega$:

$$\begin{aligned} &\text{periodic conditions on } \partial\Omega_x, \\ &v_{|y=0} = v_{|y=h} = 0, \\ &\partial_y \mathbf{u}_{|y=0} = \partial_y \mathbf{u}_{|y=h} = 0. \end{aligned} \tag{16}$$

We also assume that the distribution of the horizontal component of the velocity $\mathbf{u}$ and the density distribution are known at the initial time $t = 0$:

$$\begin{aligned} &\mathbf{u}(0, x, y) = \mathbf{u}_0(x, y), \\ &\rho(0, x, y) = \xi_0(x) e^{-g/c^2 y}. \end{aligned} \tag{17}$$

The fact that the initial condition for the density ρ has the form (17) is justified at the end of Section 2.1.
We assume that ξ_0 is a bounded positive function:

$$0 \leqslant \xi_0(x) \leqslant M < +\infty. \tag{18}$$

2.1 Formal derivation of the simplified CPEs

Taking advantages of the shallowness of the atmosphere, we assume that the characteristic scale for the altitude H is small with respect to the characteristic length L. So, the ratio of the vertical scale to the horizontal one is assumed small. In this context, we assume that the vertical movements and variations are very small compared to the horizontal ones, which justifies the following approximation.

Let ε be a "small" parameter such as:

$$\varepsilon = \frac{H}{L} = \frac{V}{U}$$

where V and U are respectively the characteristic scale of the vertical and horizontal velocity. We introduce the characteristic time T such as: $T = \dfrac{L}{U}$ and the pressure unit $P = \overline{\rho}\, U^2$ where $\overline{\rho}$ is a characteristic density. Finally, we note the dimensionless quantities of time, space, fluid velocity, pressure, density and viscosities:

$$\widetilde{t} = \frac{t}{T}, \quad \widetilde{x} = \frac{x}{L}, \quad \widetilde{y} = \frac{y}{H}, \quad \widetilde{u} = \frac{\mathbf{u}}{U}, \quad \widetilde{v} = \frac{v}{V},$$

$$\widetilde{p} = \frac{p}{\overline{\rho} U^2}, \quad \widetilde{\rho} = \frac{\rho}{\overline{\rho}}, \quad \widetilde{\lambda} = \frac{\lambda}{\overline{\lambda}}, \qquad \widetilde{\mu}_j = \frac{\mu_j}{\overline{\mu_j}}, j = 1,2,3$$

With these notations, the Froude number F_r, the Reynolds number associated to the viscosity μ_i (i=1,2,3), Re_i, the Reynolds number associated to the viscosity λ, Re_λ, and the Mach number M_a are written respectively:

$$F_r = \frac{U}{\sqrt{g\,H}}, \quad Re_i = \frac{\overline{\rho} U L}{\overline{\mu_i}}, \quad Re_\lambda = \frac{\overline{\rho} U L}{\overline{\lambda}}, \quad M_a = \frac{U}{c}. \tag{19}$$

Applying this scaling, System (11)–(14) is written:

$$\left\{\begin{array}{l}
\dfrac{1}{T}\partial_{\widetilde{t}}\widetilde{\rho} + \dfrac{U}{L}\mathrm{div}_{\widetilde{x}}\left(\widetilde{\rho}\,\widetilde{u}\right) + \dfrac{V}{H}\partial_{\widetilde{y}}\left(\widetilde{\rho v}\right) = 0, \\
\dfrac{\overline{\rho}\,U}{T}\partial_{\widetilde{t}}\left(\widetilde{\rho}\,\widetilde{u}\right) + \dfrac{\overline{\rho}\,U^2}{L}\mathrm{div}_{\widetilde{x}}\left(\widetilde{\rho}\,\widetilde{u}\otimes\widetilde{u}\right) + \dfrac{\overline{\rho}\,U\,V}{H}\partial_{\widetilde{y}}\left(\widetilde{\rho}\,\widetilde{v}\widetilde{u}\right) + \dfrac{c^2\,\overline{\rho}}{L}\nabla_{\widetilde{x}}\widetilde{\rho} = \\
\dfrac{\overline{\mu_1}\,U}{L^2}\mathrm{div}_{\widetilde{x}}\left(\mu_1 D_{\widetilde{x}}(\widetilde{u})\right) + \dfrac{\overline{\mu_2}\,U}{H^2}\partial_{\widetilde{y}}\left(\widetilde{\mu_2}\,\partial_{\widetilde{y}}\widetilde{u}\right) + \dfrac{\overline{\mu_2}\,V}{L\,H}\partial_{\widetilde{y}}\left(\widetilde{\mu_2}\,\nabla_{\widetilde{x}}\widetilde{v}\right) + \\
\dfrac{\overline{\lambda}\,U}{L^2}\nabla_{\widetilde{x}}\left(\widetilde{\lambda}\,\mathrm{div}_{\widetilde{x}}\left(\widetilde{u}\right)\right) + \dfrac{\overline{\lambda}\,V}{L\,H}\nabla_{\widetilde{x}}\left(\widetilde{\lambda}\,\partial_{\widetilde{y}}\widetilde{v}\right), \\
\dfrac{\overline{\rho}\,V}{T}\partial_{\widetilde{t}}\left(\widetilde{\rho}\,\widetilde{v}\right) + \dfrac{\overline{\rho}\,U\,V}{L}\mathrm{div}_{\widetilde{x}}\left(\widetilde{\rho}\,\widetilde{u}\,\widetilde{v}\right) + \dfrac{\overline{\rho}\,V^2}{H}\partial_{\widetilde{y}}\left(\widetilde{\rho}\,\widetilde{v}^2\right) + \dfrac{c^2\,\overline{\rho}}{H}\partial_{\widetilde{y}}\widetilde{\rho} = \\
-g\,\overline{\rho}\widetilde{\rho} + \dfrac{\overline{\mu_3}\,U}{LH}\mathrm{div}_{\widetilde{x}}\left(\widetilde{\mu_3}\,\partial_{\widetilde{y}}\widetilde{u}\right) + \dfrac{\overline{\mu_3}\,V}{L^2}\mathrm{div}_{\widetilde{x}}\left(\widetilde{\mu_3}\,\nabla_{\widetilde{x}}\widetilde{v}\right) + 2\dfrac{\overline{\mu_3}\,V}{H^2}\partial_{\widetilde{y}}(\widetilde{\mu_3}\partial_{\widetilde{y}}\widetilde{v}) \\
+\dfrac{\overline{\lambda}\,U}{L\,H}\partial_{\widetilde{y}}\left(\widetilde{\lambda}\,\mathrm{div}_{\widetilde{x}}\left(\widetilde{u}\right)\right) + \dfrac{\overline{\lambda}\,V}{H^2}\partial_{\widetilde{y}}\left(\widetilde{\lambda}\,\partial_{\widetilde{y}}\widetilde{v}\right).
\end{array}\right. \tag{20}$$

Using the definition of the dimensionless number (19), dropping $\widetilde{\cdot}$, multiplying the mass equation of System (20) by T, the momentum equation for $\mathbf{u}$ of System

(20) by $\dfrac{T}{\overline{\rho}\,U}$, the momentum equation for v of System (20) by $\dfrac{T}{\overline{\rho}\,V}$, we get the non-dimensional version of System (11)–(14) as follows:

$$\left\{\begin{array}{l}\partial_t\rho+\mathrm{div}_x\,(\rho\,\mathbf{u})+\partial_y\,(\rho v)=0,\\ \partial_t\,(\rho\,\mathbf{u})+\mathrm{div}_x\,(\rho\,\mathbf{u}\otimes\mathbf{u})+\partial_y\,(\rho\,v\mathbf{u})+\dfrac{1}{M_a^2}\nabla_x\rho=\dfrac{1}{Re_1}\mathrm{div}_x\,(\mu_1 D_x(\mathbf{u}))\\ +\dfrac{1}{Re_2}\partial_y\left(\mu_2\left(\dfrac{1}{\varepsilon^2}\partial_y\mathbf{u}+\nabla_x v\right)\right)+\dfrac{1}{Re_\lambda}\nabla_x\left(\lambda\,\mathrm{div}_x(\mathbf{u})+\lambda\partial_y v\right),\\ \partial_t\,(\rho\,v)+\mathrm{div}_x\,(\rho\,\mathbf{u}\,v)+\partial_y\left(\rho\,v^2\right)+\dfrac{1}{\varepsilon^2}\dfrac{1}{M_a^2}\partial_y\rho=-\dfrac{1}{\varepsilon^2}\dfrac{1}{F_r^2}\rho\\ +\dfrac{1}{Re_3}\mathrm{div}_x\left(\mu_3\left(\dfrac{1}{\varepsilon^2}\partial_y\mathbf{u}+\nabla_x v\right)\right)+\dfrac{2}{\varepsilon^2 Re_3}\partial_y(\mu_3\partial_y v)\\ +\dfrac{1}{\varepsilon^2\,Re_\lambda}\partial_y\left(\lambda\,\mathrm{div}_x(\mathbf{u})+\lambda\partial_y v\right).\end{array}\right. \tag{21}$$

Next, if we assume the following asymptotic regime:

$$\frac{\mu_1(\rho)}{Re_1}=\nu_1(\rho),\ \frac{\mu_i(\rho)}{Re_i}=\varepsilon^2\nu_i(\rho),\ i=2,3 \text{ and } \frac{\lambda(\rho)}{Re_\lambda}=\varepsilon^2\gamma(\rho). \tag{22}$$

and drop all terms of order $O(\varepsilon)$, System (21) reduces to the following model:

$$\left\{\begin{array}{l}\partial_t\rho+\mathrm{div}_x\,(\rho\,\mathbf{u})+\partial_y\,(\rho v)=0,\\ \partial_t\,(\rho\,\mathbf{u})+\mathrm{div}_x\,(\rho\,\mathbf{u}\otimes\mathbf{u})+\partial_y\,(\rho\,v\mathbf{u})+\dfrac{1}{M_a^2}\nabla_x p(\rho)=\mathrm{div}_x\,(\nu_1 D_x(\mathbf{u}))\\ +\partial_y\,(\nu_2\partial_y\mathbf{u})+\rho f,\\ \partial_y p(\rho)=-\dfrac{M_a^2}{F_r^2}\rho,\end{array}\right. \tag{23}$$

called *simplified CPEs.* In the sequel, we simplify by setting $M_a=F_r$. Then, the hydrostatic equation of System (23) with the pressure law (14) provides the density as

$$\rho(t,x,y)=\xi(t,x)e^{-y} \tag{24}$$

for some function $\xi=\xi(t,x)$ also called "density". Let us note that the density ρ is stratified: it means that for any altitude y, the density ρ has the profile of the function ξ. Therefore, Equation (24) justifies the choice of the initial data (17) for the density ρ at the time $t=0$. In the sequel, we also assume that:

$$\nu_i(\rho)=\nu\rho,\ i=1,2,\ \text{for } \nu>0. \tag{25}$$

2.2 The main result

Assuming the viscosity under the form (25) and $M_a=F_r$, we define:

Definition 1 *A weak solution of System (23) on $[0,T]\times\Omega$, with boundary (16)* *and initial conditions (17), is a collection of functions* $(\rho, \boldsymbol{u}, w)$*, if*

$$\begin{array}{ll} \rho \in L^\infty(0,T;L^3(\Omega)), & \sqrt{\rho} \in L^\infty(0,T;H^1(\Omega)), \\ \sqrt{\rho}\boldsymbol{u} \in L^2(0,T;L^2(\Omega)^2), & \sqrt{\rho}v \in L^\infty(0,T;(L^2(\Omega))), \\ \sqrt{\rho}D_x(\boldsymbol{u}) \in L^2(0,T;(L^2(\Omega))^{2\times 2}), & \sqrt{\rho}\partial_y v \in L^2(0,T;L^2(\Omega)), \\ \nabla\sqrt{\rho} \in L^2(0,T;(L^2(\Omega))^3) & \end{array}$$

with $\rho \geqslant 0$ *and where* $(\rho, \sqrt{\rho}\boldsymbol{u}, \sqrt{\rho}v)$ *satisfies:*

$$\left\{ \begin{array}{l} \partial_t \rho + div_x(\sqrt{\rho}\sqrt{\rho}\boldsymbol{u}) + \partial_y(\sqrt{\rho}\boldsymbol{u}\sqrt{\rho}v) = 0, \\ \rho(0,x) = \rho_0(x) \end{array} \right. \tag{26}$$

in the distribution sense, and the following equality holds for all smooth test function φ *with compact support such as* $\varphi(T,x,y) = 0$ *and* $\varphi_0 = \varphi_{t=0}$*:*

$$\begin{array}{l} -\displaystyle\int_0^T \int_\Omega \rho\boldsymbol{u}\partial_t\varphi \, dxdydt + \int_0^T \int_\Omega (2\nu\rho D_x(\boldsymbol{u}) - \rho\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\varphi \, dxdydt \\ -\displaystyle\int_0^T \int_\Omega \rho v\boldsymbol{u}\partial_y\varphi \, dxdydt - \nu\int_0^T \int_\Omega \rho\boldsymbol{u}\partial^2_{yy}\varphi \, dxdydt + \int_0^T \int_\Omega r\rho|\boldsymbol{u}|\boldsymbol{u}\varphi \, dxdydt \\ -\displaystyle\int_0^T \int_\Omega \rho div(\varphi) \, dxdzdt + \int_0^T \int_\Omega \rho v\varphi \, dxdzdt = \int_\Omega \rho_0\boldsymbol{u}_0\varphi_0 \, dxdy. \end{array} \tag{27}$$

Now, we state the main result of this paper:

Theorem 1 *Let* $(\rho_n, \boldsymbol{u}_n, v_n)$ *be a sequence of weak solutions of System (23), with boundary (16) and initial conditions (17), satisfying entropy inequalities (37) and (51) such as*

$$\rho_n \geqslant 0, \quad \rho_0^n \to \rho_0 \text{ in } L^1(\Omega), \quad \rho_0^n\boldsymbol{u}_0^n \to \rho_0\boldsymbol{u}_0 \text{ in } L^1(\Omega). \tag{28}$$

Then, up to a subsequence,

- ρ_n *converges strongly in* $\mathcal{C}^0(0,T;L^{3/2}(\Omega))$*,*
- $\sqrt{\rho_n}\boldsymbol{u}_n$ *converges strongly in* $L^2(0,T;L^{3/2}(\Omega)^2)$*,*
- $\rho_n u_n$ *converges strongly in* $L^1(0,T;L^1(\Omega)^2)$ *for all* $T > 0$*,*
- $(\rho_n, \sqrt{\rho_n}\boldsymbol{u}_n, \sqrt{\rho_n}v_n)$ *converges to a weak solution of System (26),*
- $(\rho_n, \boldsymbol{u}_n, v_n)$ *satisfy the entropy inequalities (37) and (51) and converge to a weak solution of (23)-(16).*

The proof of the main result is divided into three parts: the first part consists in writing System (23), using $(\xi, \mathbf{u}, w = e^{-y}v)$ as unknowns instead of $(\rho, \mathbf{u}, v)$. The obtained model is called *model problem* (see Section 3.1). In the second part of the proof, we show the stability of weak solutions of the model problem (see Section 3.2.2-3.2.6). In the third and last part, by a simple criterion, the main result is proved (see Section 3.2.7).

3 Proof of the main result

The first part of the proof of Theorem 1 consists in writing the simplified model (23) in a more practical way, since the standard technique fails, as pointed out in Section 1.

3.1 A model problem; an intermediate model

We first begin, by noticing that the structure of the density ρ defined as a tensorial product (see (24)) suggests the following change of variables:

$$z = 1 - e^{-y} \tag{29}$$

where the vertical velocity, in the new coordinates, is:

$$w(t,x,z) = e^{-y} v(t,x,y). \tag{30}$$

Since the new vertical coordinate z is defined as $\dfrac{d}{dy} z = e^{-y}$, multiplying by e^{y} System (23) and using the viscosity profile (25) and the change of variables (29) provides the following model:

$$\left\{\begin{array}{l}\partial_t \xi + \mathrm{div}_x\,(\xi\,\mathbf{u}) + \partial_z\,(\xi\,w) = 0,\\ \partial_t\,(\xi\,\mathbf{u}) + \mathrm{div}_x\,(\xi\,\mathbf{u}\otimes\mathbf{u}) + \partial_z\,(\xi\,\mathbf{u}\,w) + \nabla_x \xi = \nu\mathrm{div}_x\,(\xi D_x(\mathbf{u})) + \nu\partial^2_{zz}(\xi\mathbf{u}),\\ \partial_z \xi = 0.\end{array}\right. \tag{31}$$

which is the simplified CPEs (23) with the unknowns

$$(\xi(t,x), \mathbf{u}(t,x,y), w(t,x,y)) \text{ instead of } (\rho(t,x,y), \mathbf{u}(t,x,y), v(t,x,y))$$

that we call *model problem.* In the new variables, the boundary conditions (16) and the initial conditions (17) are written:

$$\begin{array}{l}\text{periodic conditions on } \Omega_x,\\ w_{|z=0} = w_{|z=h} = 0,\\ \partial_z \mathbf{u}_{|z=0} = \partial_z \mathbf{u}_{|z=h} = 0\end{array} \tag{32}$$

and

$$\begin{array}{l}\mathbf{u}(0,x,y) = \mathbf{u}_0(x,z),\\ \xi(0,x) = \xi_0(x)\end{array} \tag{33}$$

where $\Omega = \mathbb{T}^2 \times [0,h]$ with $h = 1 - e^{-1}$.

3.2 Mathematical study of the model problem

This section is devoted to the study of stability of weak solutions of System (31) and equivalently for System (23) as we will see in Section 3.2.7. In what follows, we can say that:

Definition 2 *A weak solution of System (31) on* $[0,T]\times\Omega$*, with boundary (32) and initial conditions (33), is a collection of functions* $(\xi, \boldsymbol{u}, w)$*, if*

$$\begin{array}{ll} \xi \in L^\infty(0,T;L^3(\Omega)), & \sqrt{\xi} \in L^\infty(0,T;H^1(\Omega)), \\ \sqrt{\xi}\boldsymbol{u} \in L^2(0,T;L^2(\Omega)), & \sqrt{\xi}w \in L^\infty(0,T;(L^2(\Omega))^2) \\ \sqrt{\xi}D_x(\boldsymbol{u}) \in L^2(0,T;(L^2(\Omega))^{2\times 2}), & \sqrt{\xi}\partial_z w \in L^2(0,T;L^2(\Omega)), \\ \nabla_x\sqrt{\xi} \in L^2(0,T;(L^2(\Omega))^2) & \end{array}$$

with $\xi \geqslant 0$ *and* $(\xi, \sqrt{\xi}\boldsymbol{u}, \sqrt{\xi}w)$ *satisfies:*

$$\left\{ \begin{array}{l} \partial_t \xi + div_x(\sqrt{\xi}\sqrt{\xi}\boldsymbol{u}) + \partial_z(\sqrt{\xi}\boldsymbol{u}\sqrt{\xi}w) = 0, \\ \xi(0,x) = \xi_0(x) \end{array} \right. \tag{34}$$

in the distribution sense, and the following equality holds for all smooth test function φ *with compact support such as* $\varphi(T,x,z) = 0$ *and* $\varphi_0 = \varphi_{t=0}$*:*

$$\begin{aligned} &-\int_0^T\int_\Omega \xi\boldsymbol{u}\partial_t\varphi\, dxdzdt + \int_0^T\int_\Omega (2\nu\xi D_x(\boldsymbol{u}) - \xi\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\varphi\, dxdzdt \\ &-\int_0^T\int_\Omega \xi w\boldsymbol{u}\partial_z\varphi\, dxdzdt - \nu\int_0^T\int_\Omega \xi\boldsymbol{u}\partial_{zz}^2\varphi\, dxdzdt + \int_0^T\int_\Omega r\xi|\boldsymbol{u}|\boldsymbol{u}\varphi\, dxdzdt \\ &-\int_0^T\int_\Omega \xi\, div_x(\varphi)\, dxdzdt = \int_\Omega \xi_0\boldsymbol{u}_0\varphi_0\, dxdz. \end{aligned} \tag{35}$$

We then have the following result:

Theorem 2 *Let* $(\xi_n, \boldsymbol{u}_n, w_n)$ *be a sequence of weak solutions of System (31), with boundary (32) and initial conditions (33), satisfying entropy inequalities (37) and (51) such as*

$$\xi_n \geqslant 0, \quad \xi_0^n \to \xi_0 \text{ in } L^1(\Omega), \quad \xi_0^n\boldsymbol{u}_0^n \to \xi_0\boldsymbol{u}_0 \text{ in } L^1(\Omega). \tag{36}$$

Then, up to a subsequence,

- ξ_n *converges strongly in* $\mathcal{C}^0(0,T;L^{3/2}(\Omega))$*,*
- $\sqrt{\xi_n}\boldsymbol{u}_n$ *converges strongly in* $L^2(0,T;L^{3/2}(\Omega)^2)$*,*
- $\xi_n u_n$ *converges strongly in* $L^1(0,T;L^1(\Omega)^2)$ *for all* $T > 0$*,*
- $(\xi_n, \sqrt{\xi_n}\boldsymbol{u}_n, \sqrt{\xi_n}w_n)$ *converges to a weak solution of System (34),*
- $(\xi_n, \boldsymbol{u}_n, w_n)$ *satisfy the entropy inequalities (37) and (51) and converge to a weak solution of (31)-(32).*

The proof of Theorem 2 is divided into three steps:

1. we first obtain suitable *a priori* bounds on $(\xi, \mathbf{u}, w)$ (see Section 3.2.1).

2. assuming the existence of sequences of weak solutions $(\xi_n, \mathbf{u}_n, w_n)$, we show the compactness of sequences $(\xi_n, \mathbf{u}_n, w_n)$ in apropriate space function (see Section 3.2.2-3.2.5).

3. using the obtained convergence, we show that we can pass to the limit in all terms of System (31): this finishes the proof of Theorem 2 (see Section 3.2.6).

3.2.1 Energy and entropy estimates

A part of *a priori* bounds on $(\xi, \mathbf{u}, w)$ are obtained by the physical energy inequality which is obtained in a very classical way by multiplying the momentum equation by $\mathbf{u}$, using the mass equation and integrating by parts. We obtain the following inequality:

$$\frac{d}{dt}\int_\Omega \big(\xi\frac{\mathbf{u}^2}{2} + (\xi\ln\xi - \xi + 1)\big) + \int_\Omega \xi(|D_x(\mathbf{u})|^2 + |\partial_z \mathbf{u}|^2) + r\int_\Omega \xi|\mathbf{u}|^3 \leqslant 0 \quad (37)$$

which provides the uniform estimates:

$$\sqrt{\xi}\mathbf{u} \text{ is bounded in } L^\infty(0,T;(L^2(\Omega))^2), \quad (38)$$
$$\xi^{1/3}\mathbf{u} \text{ is bounded in } L^3(0,T;(L^3(\Omega))^2), \quad (39)$$
$$\sqrt{\xi}\partial_z\mathbf{u} \text{ is bounded in } L^2(0,T;(L^2(\Omega))^2), \quad (40)$$
$$\sqrt{\xi}D_x(\mathbf{u}) \text{ is bounded in } L^2(0,T;(L^2(\Omega))^{2\times 2}), \quad (41)$$
$$\xi\ln\xi - \xi + 1 \text{ is bounded in } L^\infty(0,T;L^1(\Omega)). \quad (42)$$

As pointed out by several authors (see e.g. [3, 13]), the crucial point in the proof of the stability in these kind of models is to pass to the limit in the non linear term $\xi\mathbf{u}\otimes\mathbf{u}$ which requires the strong convergence of $\sqrt{\xi}\mathbf{u}$. So we need additional information, which may be for instance provided by the mathematical BD-entropy [2]:
we first take the gradient of the mass equation, then we multiply by 2ν and write the terms $\nabla_x\xi$ as $\xi\nabla_x\ln\xi$ to obtain:

$$\begin{aligned}&\partial_t\left(2\nu\xi\nabla_x\ln\xi\right) + \mathrm{div}_x\left(2\nu\xi\nabla_x\ln\xi\otimes\mathbf{u}\right) + \partial_z\left(2\nu\xi\nabla_x\ln\xi w\right)\\ &+\mathrm{div}_x\left(2\nu\xi\nabla_x^t\mathbf{u}\right) + \partial_z\left(2\nu\xi\nabla_x w\right) = 0.\end{aligned} \quad (43)$$

Next, we sum Equation (43) with the momentum equation of System (31) to get the equation:

$$\begin{aligned}&\partial_t\left(\xi\,\psi\right) + \mathrm{div}_x\left(\psi\otimes\xi\mathbf{u}\right) + \partial_z\left(\xi\,w\,\psi\right) + \partial_z\left(2\nu\xi\,\nabla_x w\right)\\ &+\nabla_x\xi = 2\nu\mathrm{div}_x\left(\xi A_x(\mathbf{u})\right) - r\xi|\mathbf{u}|\mathbf{u} + \nu\xi\partial_z\left(\partial_z\mathbf{u}\right)\end{aligned} \quad (44)$$

where $\psi = \mathbf{u} + 2\nu\nabla_x\ln\xi$ and $2A_x(\mathbf{u}) = \nabla_x\mathbf{u} - \nabla_x^t\mathbf{u}$ is the vorticity tensor. The mathematical BD-entropy is then obtained by multiplying the previous equation by ψ and integrating by parts:

- $$\int_\Omega \left(\partial_t\left(\xi\,\psi\right) + \mathrm{div}_x\left(\psi\otimes\xi\mathbf{u}\right) + \partial_z\left(\xi\,w\,\psi\right)\right)\psi\,dxdz = \frac{d}{dt}\int_\Omega \xi\frac{|\psi|^2}{2}\,dxdz. \tag{45}$$

- Since
$$\int_\Omega 2\nu\mathrm{div}_x\left(\xi A_x(\mathbf{u})\right)\nabla_x\ln\xi\,dxdz = 0$$
and periodic boundary conditions are assumed on Ω_x, we have:
$$\int_\Omega 2\nu\mathrm{div}_x\left(\xi A_x(\mathbf{u})\right)\mathbf{u}\,dxdz = 2\nu\int_\Omega \xi|A_x(\mathbf{u})|^2\,dxdz. \tag{46}$$

- Derivating the mass equation with respect to z provides the identity
$$\partial_z\mathrm{div}_x(\xi\mathbf{u}) = -\xi\partial^2_{zz}w$$
and also recalling that ξ is only x-dependent, the integral becomes:
$$\begin{aligned}\int_\Omega \partial_z\left(2\nu\xi\,\nabla_x w\right)\psi\,dxdz &= \int_\Omega \partial_z\left(2\nu\xi\,\nabla_x w\right)\mathbf{u}\,dxdz\\ &= \int_\Omega w\partial_z\mathrm{div}_x(\xi\mathbf{u})\,dxdz\\ &= \int_\Omega \xi|\partial_z w|^2\,dxdz.\end{aligned} \tag{47}$$

- The other terms are easily computed. We have:
$$\int_\Omega r\xi|\mathbf{u}|\mathbf{u}\psi\,dxdz = \int_\Omega r|\mathbf{u}|^3\,dxdz + \int_\Omega 2\nu r|\mathbf{u}|\mathbf{u}\nabla_x\xi\,dxdz, \tag{48}$$
$$\int_\Omega \nu\partial_z\left(\partial_z\mathbf{u}\right)\psi\,dxdz = \int_\Omega \nu\xi|\partial_z\mathbf{u}|^2\,dxdz. \tag{49}$$
and
$$\begin{aligned}\int_\Omega \nabla_x\xi\psi\,dxdz &= \int_\Omega \nabla_x\xi\mathbf{u}\,dxdz + 2\nu\int_\Omega \nabla_x\xi\nabla_x\ln\xi\,dxdz\\ &= \frac{d}{dt}\int_\Omega (\xi\ln\xi - \xi + 1)\,dxdz + 8\nu\int_\Omega |\nabla_x\sqrt{\xi}|\,dxdz.\end{aligned} \tag{50}$$

Finally, gathering results (45)–(50) provides the following mathematical entropy equality:
$$\begin{aligned}&\frac{d}{dt}\int_\Omega (\xi\frac{|\psi|^2}{2} + \xi\ln\xi - \xi + 1)\,dxdz\\ &+ \int_\Omega (2\nu\xi|\partial_z w|^2 + 2\nu\xi|A_x(\mathbf{u})|^2 + \nu\xi|\partial_z\mathbf{u}|^2)\,dxdz\\ &+ \int_\Omega (r\xi|\mathbf{u}|^3 + 2\nu r|\mathbf{u}|\mathbf{u}\nabla\xi + 8\nu|\nabla_x\sqrt{\xi}|^2)\,dxdz = 0\end{aligned} \tag{51}$$

and estimates:

$$\nabla\sqrt{\xi} \text{ is bounded in } L^\infty(0,T;(L^2(\Omega))^3), \tag{52}$$
$$\sqrt{\xi}\partial_z w \text{ is bounded in } L^2(0,T;(L^2(\Omega))), \tag{53}$$
$$\sqrt{\xi}A_x(\mathbf{u}) \text{ is bounded in } L^2(0,T;(L^2(\Omega))^{2\times 2}). \tag{54}$$

which finishes the first step of the proof of Theorem 2.

Remark 2 *Estimate (52) is a straightforward consequence of estimates* $\sqrt{\xi}\psi \in L^\infty(0,T,L^2(\Omega)^2)$ *and* $\sqrt{\xi}\boldsymbol{u} \in L^\infty(0,T,L^2(\Omega)^2)$ *since*

$$\sqrt{\xi}\psi = \sqrt{\xi}\boldsymbol{u} + \frac{\nabla_x\xi}{\sqrt{\xi}}.$$

The second step of the proof of Theorem 2 will be divided into 4 parts. The first part consists to show the convergence of $\sqrt{\xi_n}$ (see Section 3.2.2). Then, we seek for bounds of $\sqrt{\xi_n}\mathbf{u}_n$ and $\sqrt{\xi_n}w_n$ in an appropriate space (see Section 3.2.3) to be able to prove the convergence of $\xi_n\mathbf{u}_n$ (see Section 3.2.4). Thereafter, the convergence of $\sqrt{\xi_n}\mathbf{u}_n$ is obtained (see Section 3.2.5).

3.2.2 Convergence of $\sqrt{\xi_n}$

The first part of the proof of Theorem 2 consists to show the following convergence.

Lemma 1 *For every* ξ_n *satisfying the mass equation of System (31), we have:*

$$\sqrt{\xi_n} \textit{ is bounded in } L^\infty(0,T,H^1(\Omega)),$$
$$\partial_t\sqrt{\xi_n} \textit{ is bounded in } L^2(0,T,H^{-1}(\Omega)).$$

Then, up to a subsequence, the sequence ξ_n *converges almost everywhere and strongly in* $L^2(0,T;L^2(\Omega))$. *Moreover,* ξ_n *converges to* ξ *in* $\mathcal{C}^0(0,T;L^{3/2}(\Omega))$.

Proof of Lemma 1:

The mass conservation equation gives

$$||\sqrt{\xi_n}(t)||^2_{L^2(\Omega)} = ||\xi_0^n||_{L^1(\Omega)}.$$

This equation and Estimate (52) give the bound of $\sqrt{\xi_n}$ in $L^\infty(0,T,H^1(\Omega))$. Using again the mass conservation equation, we write

$$\begin{aligned}\partial_t(\sqrt{\xi_n}) &= -\frac{1}{2}\sqrt{\xi_n}\mathrm{div}_x(\mathbf{u}_n) - \mathbf{u}_n.\nabla_x\sqrt{\xi_n} - \sqrt{\xi_n}\partial_z w_n \\ &= \frac{1}{2}\sqrt{\xi_n}\mathrm{div}_x(\mathbf{u}_n) - \mathrm{div}_x(\mathbf{u}_n\sqrt{\xi_n}) - \sqrt{\xi_n}\partial_z w_n.\end{aligned}$$

Then from Estimates (41), (54), (52), (53),

$$\partial_t\sqrt{\xi_n} \text{ is bounded in } L^2(0,T,H^{-1}(\Omega)).$$

Next, Aubin's lemma provides compactness of $\sqrt{\xi_n}$ in $\mathcal{C}^0(0,T,L^2(\Omega)$, that is:

$$\sqrt{\xi_n} \text{ converges strongly to } \sqrt{\xi} \text{ in } \mathcal{C}^0(0,T,L^2(\Omega)). \tag{55}$$

We also have, by Sobolev embeddings, bounds of $\sqrt{\xi_n}$ in spaces $L^\infty(0,T,L^p(\Omega))$ for all $p \in [1,6]$.
Consequently, for $p=6$, we get bounds of ξ_n in $L^\infty(0,T,L^3(\Omega))$ and we deduce that

$$\xi_n \mathbf{u}_n = \sqrt{\xi_n}\sqrt{\xi_n}\mathbf{u}_n \text{ is bounded in } L^\infty(0,T,L^{3/2}(\Omega)^2). \tag{56}$$

It follows that $\partial_t \xi_n$ is bounded in $L^\infty(0,T,W^{-1,3/2}(\Omega))$ since

$$\partial_t \xi_n = -\mathrm{div}(\xi_n \mathbf{u}_n) - \xi_n \partial_z w_n$$

and we have Estimate (53).

To conclude, writing

$$\nabla_x \xi_n = 2\sqrt{\xi_n}\nabla_x \sqrt{\xi_n} \in L^\infty(0,T;L^{3/2}(\Omega)^2),$$

we deduce bounds of ξ_n in $L^\infty(0,T;W^{1,3/2}(\Omega))$. Then Aubin's lemma provides compactness of ξ_n in the intermediate space $L^{3/2}(\Omega)$:

$$\text{compactness of } \xi_n \text{ in } \mathcal{C}^0(0,T;L^{3/2}(\Omega)).$$

■

3.2.3 Bounds of $\sqrt{\xi_n}\mathbf{u}_n$ and $\sqrt{\xi_n}w_n$

To prove the convergence of the momentum equation, we have to control bounds of $\sqrt{\xi_n}\mathbf{u}_n$ and $\sqrt{\xi_n}w_n$.

Lemma 2 *We have*

$$\sqrt{\xi_n}\boldsymbol{u}_n \textit{ bounded in } L^\infty(0,T;(L^2(\Omega))^2)$$

and

$$\sqrt{\xi}w_n \textit{ bounded in } L^2(0,T;L^2(\Omega)).$$

Proof of Lemma 2: We have already bounds of $\sqrt{\xi_n}$ (see Estimates (38)). There is left to show bounds of $\sqrt{\xi_n}w_n$ $L^2(0,T;L^2(\Omega))$. As $\xi_n = \xi_n(t,x)$ and Estimates (53) holds, by the Poincaré inequality, we have:

$$\int_0^h |\sqrt{\xi_n}w_n|^2\,dz \leqslant c\int_0^h |\partial_z(\sqrt{\xi_n}w_n)|^2\,dz.$$

Hence,

$$\int_\Omega \xi_n|w_n|^2\,dxdz \leqslant c\int_\Omega \xi_n|\partial_z w_n|^2\,dxdz$$

gives bounds of $\sqrt{\xi_n}w_n$ in $L^2(0,T;L^2(\Omega))$.

■

3.2.4 Convergence of $\xi_n \mathbf{u}_n$

As bounds of $\sqrt{\xi_n}\mathbf{u}_n$ and $\sqrt{\xi_n}w_n$ are provided by Lemma 2, we are able to show the convergence of the momentum equation.

Lemma 3 *Let $m_n = \xi_n \boldsymbol{u}_n$ be a sequence satisfying the momentum equation (31). Then we have:*

$$\xi_n \boldsymbol{u}_n \to m \quad \textit{in } L^2(0,T;(L^p(\Omega))^2) \textit{ strong }, \forall\, 1 \leqslant p < 3/2$$

and

$$\xi_n \boldsymbol{u}_n \to m \quad \textit{a.e. } (t,x,y) \in (0,T)\times\Omega.$$

Proof of Lemma 3:
Writing $\nabla_x(\xi_n \mathbf{u}_n)$ as:

$$\nabla_x(\xi_n \mathbf{u}_n) = \sqrt{\xi_n}\sqrt{\xi_n}\nabla_x \mathbf{u}_n + 2\sqrt{\xi_n}\mathbf{u}_n \otimes \nabla\sqrt{\xi_n}$$

provides

$$\nabla_x(\xi_n \mathbf{u}_n) \text{ bounded in } L^2(0,T;(L^1(\Omega))^{2\times 2}). \tag{57}$$

Next, we have

$$\partial_z(\xi_n \mathbf{u}_n) = \sqrt{\xi_n}\sqrt{\xi_n}\partial_z(\mathbf{u}_n) \text{ is bounded } L^2(0,T;(L^{3/2}(\Omega))^2). \tag{58}$$

Then, from bounds (57) and (58), we deduce:

$$\xi_n \mathbf{u}_n \text{ is bounded } L^2(0,T;W^{1,1}(\Omega)^2). \tag{59}$$

On the other hand, we have:

$$\begin{aligned}\partial_t(\xi_n\, \mathbf{u}_n) \quad &= \quad -\mathrm{div}_x\,(\xi_n\, \mathbf{u}_n \otimes \mathbf{u}_n) - \partial_z\,(\xi_n\, \mathbf{u}_n\, w_n) - \nabla_x \xi_n \\ &\quad\quad + \nu\,\mathrm{div}_x\,(\xi_n D_x(\mathbf{u}_n)) + \nu\partial_z\,(\xi_n \partial_z \mathbf{u}_n)\,.\end{aligned}$$

As

$$\xi_n \mathbf{u}_n \otimes \mathbf{u}_n = \sqrt{\xi}\mathbf{u}_n \otimes \sqrt{\xi}\mathbf{u}_n, \tag{60}$$

we deduce bounds of

$$\xi_n \mathbf{u}_n \otimes \mathbf{u}_n \text{ in } L^\infty(0,T;(L^1(\Omega))^{2\times 2}).$$

Particularly, we have

$$\mathrm{div}(\xi_n \mathbf{u}_n \otimes \mathbf{u}_n) \text{ bounded in } L^\infty(0,T;(W^{-2,4/3}(\Omega))^2).$$

Similarly, as $\xi_n \mathbf{u}_n w_n = \sqrt{\xi}\mathbf{u}_n\sqrt{\xi}w_n \in (L^1(\Omega))^2$, we also have

$$\partial_z(\xi_n \mathbf{u}_n w_n) \text{ bounded in } L^\infty(0,T;(W^{-2,4/3}(\Omega))^2).$$

Moreover, as

$$\sqrt{\xi_n}\sqrt{\xi_n}\partial_z \mathbf{u}_n, \in L^2(0,T;(L^{3/2}(\Omega))^2) \text{ and}$$

$$\sqrt{\xi_n}\sqrt{\xi_n}D_x(\mathbf{u}_n) \in L^2(0,T;(L^{3/2}(\Omega))^{2\times 2}),$$

we get bounds of

$$\partial_z(\sqrt{\xi_n}\sqrt{\xi_n}\partial_z\mathbf{u}_n),\ \mathrm{div}_x(\sqrt{\xi_n}\sqrt{\xi_n}D_x(\mathbf{u}_n)) \in L^2(0,T;(W^{-1,3/2}(\Omega))^2).$$

We also have bounds of $\nabla_x\xi_n \in L^\infty(0,T,(W^{-1,3/2}(\Omega))^2)$.
Using $W^{-1,3/2}(\Omega) \subset W^{-1,4/3}(\Omega)$, we obtain

$$\partial_t(\xi_n\mathbf{u}_n) \text{ bounded in } L^2(0,T;W^{-2,4/3}(\Omega)^2). \tag{61}$$

Using bounds (59), (61) with Aubin's lemma provides compactness of

$$\xi_n\mathbf{u}_n \in L^2(0,T;(L^p(\Omega))^2),\ \forall p \in\ [1,3/2[. \tag{62}$$

■

3.2.5 Convergence of $\sqrt{\xi_n}\mathbf{u}_n$ and $\xi_n w_n$

Let us note that, up to Section 3.2.4, we can always define $\mathbf{u} = m/\xi$ on the set $\{\xi > 0\}$, but we do not know, *a priori*, if m equals zero on the vacuum set. To this end, we need to prove the following lemma:

Lemma 4

1. *The sequence $\sqrt{\xi_n}\boldsymbol{u}_n$ satisfies*
   - *$\sqrt{\xi_n}\boldsymbol{u}_n$ converges strongly in $L^2(0,T;L^2(\Omega))$ to $\dfrac{m}{\sqrt{\xi}}$.*
   - *We have $m = 0$ almost everywhere on the set $\{\xi = 0\}$ and there exists a function $\boldsymbol{u}$ such that $m = \xi\boldsymbol{u}$ and*

$$\xi_n\boldsymbol{u}_n \to \xi\boldsymbol{u} \text{ strongly in } L^2(0,T;L^p(\Omega)^2) \text{ for all } p \in [1,3/2[, \tag{63}$$

$$\sqrt{\xi_n}\boldsymbol{u}_n \to \sqrt{\xi}\boldsymbol{u} \text{ strongly in } L^2(0,T;L^2(\Omega)^2). \tag{64}$$

2. *The sequence $\sqrt{\xi_n}w_n$ converges weakly in $L^2(0,T;L^2(\Omega))$ to $\sqrt{\xi}w$.*

Proof of Lemma 4:
We refer to [14] for details of the first part of the proof. The second part of the theorem is done by weak compactness. As $\sqrt{\xi_n}w_n$ is bounded in $L^2(0,T;L^2(\Omega))$, there exists, up to a subsequence, $\sqrt{\xi_n}w_n$ which converges weakly some limit l in $L^2(0,T;L^2(\Omega))$. Next, we define w

$$w = \begin{cases} \dfrac{l}{\sqrt{\xi}} & \text{if} \quad \xi > 0, \\ 0 \text{ a.e.} & \text{if} \quad \xi = 0 \end{cases}$$

where the limit l is written: $l = \sqrt{\xi}\dfrac{l}{\sqrt{\xi}} = \sqrt{\xi}w$.

■

This finishes the second step of the proof of Theorem 2.

In the third and last step (see Section 3.2.6), using the convergence of Sections 3.2.2–3.2.5, we show that we can pass to the limit for all terms of System (31).

3.2.6 Convergence step

We are now ready to prove that we can pass to the limit in all terms of System (31) in the sense of Theorem 2. To this end, let $(\xi_n,\, \mathbf{u}_n,\, w_n)$ be a weak solution of System (31) satisfying Lemma 1 to 4 and let $\phi \in \mathcal{C}_c^\infty([0,T]\times\Omega)$ be a smooth function with compact support such as $\phi(T,x,z)=0$. Then, writing each term of the weak formulation of System (31), we have:

- $$\begin{aligned}\int_0^T\int_\Omega \partial_t(\xi_n\mathbf{u}_n)\phi\,dxdzdt &= -\int_0^T\int_\Omega \xi_n\mathbf{u}_n\partial_t\phi\,dxdzdt \\ &\quad -\int_\Omega \xi_0^n\mathbf{u}_0^n\phi(0,x,z)\,dxdz.\end{aligned} \tag{65}$$

  Using convergences (36) and Lemma 3, we get

  $$-\int_0^T\int_\Omega \xi_n\mathbf{u}_n\partial_t\phi\,dxdzdt - \int_\Omega \xi_0^n\mathbf{u}_0^n\phi(0,x,z)\,dxdz \to$$
  $$-\int_0^T\int_\Omega \xi\mathbf{u}\partial_t\phi\,dxdzdt - \int_\Omega \xi_0\mathbf{u}_0\phi(0,x,y)\,dxdz.$$

- $$\int_0^T\int_\Omega \mathrm{div}_x(\xi_n\mathbf{u}_n\otimes\mathbf{u}_n)\cdot\phi\,dxdzdt = -\int_0^T\int_\Omega \xi_n\mathbf{u}_n\otimes\mathbf{u}_n : \nabla_x\phi\,dxdzdt.$$

  From Equality (60) and Lemma 4, we have:

  $$-\int_0^T\int_\Omega \xi_n\mathbf{u}_n\otimes\mathbf{u}_n : \nabla_x\phi\,dxdzdt \to -\int_0^T\int_\Omega \xi\mathbf{u}\otimes\mathbf{u} : \nabla_x\phi\,dxdzdt.$$

- $$\int_0^T\int_\Omega \partial_z(\xi_n\mathbf{u}_n w_n)\cdot\phi\,dxdzdt = -\int_0^T\int_\Omega \xi_n\mathbf{u}_n w_n\cdot\partial_z\phi\,dxdzdt.$$

  As $\xi_n\mathbf{u}_n w_n = \sqrt{\xi_n}\mathbf{u}_n\sqrt{\xi_n}w_n$, by Lemma 4, we get:

  $$-\int_0^T\int_\Omega \xi_n\mathbf{u}_n w_n\cdot\partial_z\phi\,dxdzdt \to -\int_0^T\int_\Omega \xi\mathbf{u}w\cdot\partial_z\phi\,dxdzdt.$$

- $$\int_0^T\int_\Omega \nabla_x\xi_n\cdot\phi\,dxdzdt = -\int_0^T\int_\Omega \xi_n\mathrm{div}_x(\phi)\,dxdzdt.$$

  Then, Lemma 1 provides:

  $$-\int_0^T\int_\Omega \xi_n\mathrm{div}_x(\phi)\,dxdzdt \to -\int_0^T\int_\Omega \xi\mathrm{div}_x(\phi)\,dxdzdt$$

- $$\int_0^T \int_\Omega \mathrm{div}_x(\xi_n D_x(\mathbf{u}_n)) \cdot \phi \, dxdzdt = -\int_0^T \int_\Omega \xi_n D_x(\mathbf{u}_n) : \nabla \phi \, dxdzdt.$$

  Since $D_x(\mathbf{u}_n) = \dfrac{1}{2}(\nabla_x \mathbf{u}_n + \nabla_x^t \mathbf{u}_n)$, expanding the term in the last integral gives:

  $$\begin{aligned}
  &-\int_0^T \int_\Omega \xi_n D_x(\mathbf{u}_n) : \nabla_x \phi \, dxdzdt \\
  &= \frac{1}{2}\int_0^T \int_\Omega (\xi_n \mathbf{u}_n \cdot \Delta_x \phi + \nabla_x \phi \nabla_x(\sqrt{\xi_n}) \cdot \sqrt{\xi_n}\mathbf{u}_n \, dxdzdt \\
  &+ \frac{1}{2}\int_0^T \int_\Omega (\xi_n \mathbf{u}_n \cdot \mathrm{div}_x(\nabla_x^t \phi) + \nabla_x^t \sqrt{\xi_n} \cdot \nabla_x \phi \cdot \sqrt{\xi_n}\mathbf{u}_n) \, dxdzdt.
  \end{aligned}$$

  From Estimates (52), the sequence $\nabla_x \sqrt{\xi_n}$ weakly converges, and using Lemma 1, Lemma 3 and 4, we obtain:

  $$\frac{1}{2}\int_0^T \int_\Omega (\xi_n \mathbf{u}_n \cdot \Delta_x \phi + \nabla_x \phi \nabla_x(\sqrt{\xi_n}) \cdot \sqrt{\xi_n}\mathbf{u}_n \, dxdzdt$$

  $$+\frac{1}{2}\int_0^T \int_\Omega (\xi_n \mathbf{u}_n \cdot \mathrm{div}_x(\nabla_x^t \phi) + \nabla_x^t \sqrt{\xi_n} \cdot \nabla_x \phi \cdot \sqrt{\xi_n}\mathbf{u}_n) \, dxdzdt \to$$

  $$\frac{1}{2}\int_0^T \int_\Omega (\xi \mathbf{u} \cdot \Delta_x \phi + \nabla_x \phi \nabla_x(\sqrt{\xi}) \cdot \sqrt{\xi}\mathbf{u} \, dxdzdt$$

  $$+\frac{1}{2}\int_0^T \int_\Omega (\xi \mathbf{u} \cdot \mathrm{div}_x(\nabla_x^t \phi) + \nabla_x^t \sqrt{\xi} \cdot \nabla_x \phi \cdot \sqrt{\xi}\mathbf{u}) \, dxdzdt.$$

  Hence

  $$-\int_0^T \int_\Omega \xi_n D_x(\mathbf{u}_n) : \nabla_x \phi \, dxdzdt \to -\int_0^T \int_\Omega \xi D_x(\mathbf{u}) : \nabla_x \phi \, dxdzdt.$$

- $$\int_0^T \int_\Omega \partial_{zz}^2(\xi_n \mathbf{u}_n) \cdot \phi \, dxdzdt \to \int_0^T \int_\Omega \xi_n \mathbf{u}_n \cdot \partial_{zz}^2(\phi) \, dxdzdt.$$

  Using Lemma 3 provides the following convergence:

  $$\int_0^T \int_\Omega \xi_n \mathbf{u}_n \cdot \partial_{zz}^2(\phi) \, dxdzdt \to \int_0^T \int_\Omega \xi \mathbf{u} \cdot \partial_{zz}^2(\phi) \, dxdzdt$$

- $$\int_0^T \int_\Omega r\xi_n |\mathbf{u}_n| \mathbf{u}_n \cdot \phi \, dxdzdt \to \int_0^T \int_\Omega r\xi |\mathbf{u}| \mathbf{u} \cdot \phi \, dxdzdt$$

  with Lemma 4, which finishes the proof of Theorem 2.

■

3.2.7 End of the proof of Theorem 1

In order to conlude, let $(\xi_n,\ \mathbf{u}_n,\ w_n)$ be a weak solution of System (31), then all estimates 3.2.2-3.2.6 hold if we replace ξ_n by ρ_n and w_n by v_n (see [9]), since $\rho(t,x,y)=\xi(t,x)e^{-y}$ and $w(t,x,z)=v(t,xy)e^{-y}$ where $\dfrac{d}{dy}z=e^{-y}$. This proves Theorem 1.

■

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